JOURNAL OF FUNCTIONAL ANALYSIS 5, 66–70 (1970)

On Spectral Concentration for Semi-Bounded Operators*

W. M. GREENLEE

Northwestern University, Evanston, Illinois 60201

Communicated by Tosio Kato

Received November 15, 1968

Let H be a complex Hilbert space with inner product (v, w) and norm $|v|$. If the resolvent of a self adjoint operator A_ϵ $(\epsilon > 0)$ converges strongly as $\epsilon \downarrow 0$ to the resolvent of a self-adjoint operator B, and if λ is an isolated eigenvalue of B of finite multiplicity m, then A_ϵ need not have an eigenvalue near λ. In some cases, however, the spectrum of A_ϵ becomes "concentrated" near λ as ϵ is reduced. Precisely (cf. Riddell [1] or Conley and Rejto [2]), if J is an isolating interval for λ, then the spectrum of A_ϵ in J is concentrated to order $p \geqslant 0$ if there are sets $C_\epsilon \subset J$ with Lebesgue measure $o(\epsilon^p)$ as $\epsilon \downarrow 0$ such that the spectral projection assigned to A_ϵ by C_ϵ converges strongly as $\epsilon \downarrow 0$ to the eigenprojection, P, on the λ-eigenspace. According to [1], Theorem 2.7 (cf. also [2], Lemma 2.1) the spectrum of A_ϵ in J is concentrated to order p if and only if there exist m pairs $(\lambda_{j\epsilon}, u_{j\epsilon})$, $\lambda_{j\epsilon}$ complex, $u_{j\epsilon}$ in the domain of A_ϵ, $j = 1,\ldots, m$, such that for each j and $k \neq j$, $|u_{j\epsilon}| = 1$, $|(A_\epsilon - \lambda_{j\epsilon}) u_{j\epsilon}| = o(\epsilon^p)$ as $\epsilon \downarrow 0$, $|(I - P)u_{j\epsilon}| \to 0$ as $\epsilon \downarrow 0$, and $(u_{j\epsilon}, u_{k\epsilon}) \to 0$ as $\epsilon \downarrow 0$.

The concern of this note is to present criteria for concentration to order $0 < p < 1$ in the event that the formal perturbation method is not applicable for construction of the pairs $(\lambda_{j\epsilon}, u_{j\epsilon})$ (the so-called "divergence difficulty," Kato [3], §12). The lemma to be proven below includes Lemma 3.3 of [1].

So let $b(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(b)$ which is dense in H. Further assume that the corresponding quadratic form $b(v) \equiv b(v, v)$ is closed and has a positive lower bound which, without loss of generality, may be assumed to be not less than 1,

$$b(v) \geqslant (v, v), \qquad v \in D(b).$$

* This work was supported in part by N.S.F. Grant GP-8217.

Since $b(v)$ is closed, $D(b)$, with the inner product

$$(v, w)_0 = b(v, w),$$

is a Hilbert space, V_0, with norm $| v |_0$.

Further let $a(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(a)$ which is dense in V_0 (and therefore in H). The corresponding quadratic form $a(v) \equiv a(v, v)$ is assumed to be nonnegative,

$$a(v) \geqslant 0, \qquad v \in D(a),$$

and closed in V_0.

To the form $b(v, w)$ there corresponds a positive definite self adjoint operator B in H which is defined on

$$D(B) = \{v \in V_0 : w \to b(v, w) \text{ is continuous on } V_0 \text{ in the topology of } H\},$$

by

$$(Bv, w) = b(v, w), \quad v \in D(B), \quad Bv \in H, \quad w \in V_0 .$$

$D(B)$, with the inner product (Bv, Bw), is a Hilbert space. Similarly for $\epsilon > 0$ there corresponds to the form $\epsilon a(v, w) + b(v, w)$ a positive definite self adjoint operator A_ϵ in H with domain

$$D(A_\epsilon) = \{v \in D(a) : w \to \epsilon a(v, w) + b(v, w) \text{ is continuous on } D(a)$$
$$\text{in the topology of } H\},$$

by

$$(A_\epsilon v, w) = \epsilon a(v, w) + b(v, w), \quad v \in D(A_\epsilon), \quad A_\epsilon v \in H, \quad w \in D(a).$$

The concentration lemma will be formulated via the introduction of another operator which is associated with this problem. Let $\mathcal{A}$ be the nonnegative self adjoint operator in V_0 defined on

$$D(\mathcal{A}) = \{v \in D(a) : w \to a(v, w) \text{ is continuous on } D(a) \text{ in the topology of } V_0\},$$

by

$$b(\mathcal{A}v, w) = a(v, w), \quad v \in D(\mathcal{A}), \quad \mathcal{A}v \in V_0, \quad w \in D(a).$$

Then $D(\mathcal{A})$, with the inner product

$$(v, w)_1 = b(v, w) + b(\mathcal{A}v, \mathcal{A}w),$$

is a Hilbert space, V_1, with norm $| v |_1$. For $\epsilon > 0$, $B^{-1}A_\epsilon \subset \epsilon\mathcal{A} + I$ (cf. Kato [4], Corollary 2.4, p. 323).

Since $\mathcal{A}$ is a nonnegative self adjoint operator in V_0, $\mathcal{A}$ has nonnegative self adjoint fractional powers, $\mathcal{A}^\tau$, defined for $\tau \geqslant 0$ by use of the spectral theorem. Moreover, for $\tau \geqslant 0$, $D(\mathcal{A}^\tau) = D(S^\tau)$ where $S = (\mathcal{A}^2 + I)^{1/2}$. Now $(v, w)_1 = (Sv, Sw)_0$ for all $v, w \in V_1$ and so for $0 < \tau < 1$, $D(\mathcal{A}^\tau)$, with the inner product

$$(v, w)_\tau = (S^\tau v, S^\tau w)_0 ,$$

is a Hilbert space, V_τ, with corresponding norm $| v |_\tau$. V_τ is called the τth interpolation space by quadratic interpolation between V_1 and V_0 (cf. Lions [5]).

LEMMA. *Let λ be an isolated eigenvalue of B of multiplicity $m < \infty$ with corresponding eigenprojection P, and let J be an isolating interval for λ.* (i) *If $0 \leqslant \tau < 1$ and $PH \subset V_\tau$, then the spectrum of A_ϵ in J is concentrated to order τ. If in addition $D(B) \subset_c V_\sigma$* (i.e., $D(B) \subset V_\sigma$ with continuous injection) *for fixed $\sigma > 0$, then the spectrum of A_ϵ in J is concentrated to order $\tau + \sigma$ if $\tau + \sigma < 1$ and to order p for all $p < 1$ if $\tau + \sigma \geqslant 1$.* (ii) *If $PH \subset V_1$, then the spectrum of A_ϵ in J is concentrated to order p for all $p < 1$.*

Proof. Let $\tau \in [0, 1]$ and $u \in PH \subset V_\tau$. Then since $Bu = \lambda u$ and $A_\epsilon^{-1} B \subset (\epsilon\mathcal{A} + I)^{-1}$,

$$|(A_\epsilon - \lambda) A_\epsilon^{-1} u |^2 = |[I - (\epsilon\mathcal{A} + I)^{-1}] u |^2 \leqslant |[I - (\epsilon\mathcal{A} + I)^{-1}] u |_0^2 . \quad (1)$$

Letting E be the resolution of the identity for the self-adjoint operator $\mathcal{A}$, the spectral theorem for functions of a self-adjoint operator gives,

$$|[I - (\epsilon\mathcal{A} + I)^{-1}] u |_0^2 = \int_0^\infty [1 - (\epsilon\mu + 1)^{-1}]^2 (E(d\mu) u, u)_0$$

$$= \epsilon^{2\tau} \int_0^\infty \mu^{2\tau} \frac{(\epsilon\mu)^{2-2\tau}}{(\epsilon\mu + 1)^{2-2\tau}} \cdot \frac{1}{(\epsilon\mu + 1)^{2\tau}} (E(d\mu) u, u)_0 . \quad (2)$$

Since $u \in D(\mathcal{A}^\tau) = V_\tau$ if and only if $\int_0^\infty \mu^{2\tau}(E(d\mu) u, u)_0 < \infty$, it follows from the dominated convergence theorem that

$$| u - \lambda A_\epsilon^{-1} u | = |(A_\epsilon - \lambda) A_\epsilon^{-1} u | = \begin{cases} o(\epsilon^\tau) & \text{as } \epsilon \downarrow 0, \quad \tau \in [0, 1), \\ O(\epsilon) & \text{as } \epsilon \downarrow 0, \quad \tau = 1. \end{cases} \quad (3)$$

Now let $\{u_1, ..., u_m\}$ be an orthonormal basis for PH and $u_{j\epsilon} = | A_\epsilon^{-1} u_j |^{-1} A_\epsilon^{-1} u_j$, $j = 1, ..., m$. Then application of Theorem 2.7, [1], to the pairs $(\lambda, u_{j\epsilon})$, $j = 1, ..., m$, yields concentration to order τ of the spectrum of A_ϵ in J if $\tau \in [0, 1)$ and concentration to order p for all $p < 1$ if $\tau = 1$. So (ii) and the first part of (i) are proven.

If now $D(B) \subset_c V_\sigma$ then by Corollary 3.1, Greenlee [6], there exists a constant $M > 0$ such that $| v | \leqslant M | S^{-\sigma} v |_0$ for all $v \in V_0$. Thus in place of (1) and (2) one has for $u \in PH$,

$$
\begin{aligned}
|(A_\epsilon - \lambda) A_\epsilon^{-1} u|^2 &\leqslant M^2 | S^{-\sigma}[I - (\epsilon \mathcal{O}\!\!\ell + I)^{-1}] u |_0^2 \\
&= M^2 \epsilon^{2\tau+2\sigma} \int_0^\infty \mu^{2\tau+2\sigma} \frac{(\epsilon\mu)^{2-2\tau-2\sigma}}{(\epsilon\mu + 1)^{2-2\tau-2\sigma}} \\
&\quad \cdot \frac{1}{(\epsilon\mu + 1)^{2\tau+2\sigma}} (\mu^2 + 1)^{-2\sigma} (E(d\mu) u, u)_0 ,
\end{aligned}
$$

and the remaining conclusion again follows from the dominated convergence theorem.

Remarks. Observe that the rate of convergence estimates obtained in the lemma are actually obtained with (possibly) stronger norms than needed for the assertions on spectral concentration. As noted in Kato [7], §6, it is possible to replace the condition $b(v) \geqslant | v |^2$ by $b(v) \geqslant -\delta | v |^2$, $\delta \geqslant 0$. One considers $b'(v) = b(v) + (\delta + 1) | v |^2$ which amounts to a change of origin for all spectra. Similarly the condition $a(v) \geqslant 0$ can be replaced by $a(v) \geqslant -\eta | v |^2 - \theta b(v)$; $\eta, \theta \geqslant 0$, by considering $a'(v) = a(v) + \eta | v |^2 + \theta b(v)$.

EXAMPLE. (cf. Rellich [8] and Friedrichs and Rejto [9]). Let H be $L^2(0, \infty)$ with the usual inner product and let B be the integral operator given by

$$Bv(x) = k(x) \int_0^\infty \overline{k(y)}\, v(y)\, dy = k(x)(v, k)$$

where k is continuous on $[0, \infty)$, of norm 1, and $k(x) \neq 0$ for $x \in [0, \infty)$ (recall the preceding remark). The self adjoint operator B has the eigenvalue 1 with eigenfunction $k(x)$ and the eigenvalue zero with infinite multiplicity.

Let the perturbing form be

$$a(v) = \int_0^\infty x \, | v(x)|^2 \, dx$$

defined for all $v \in L^2(0, \infty)$ such that $x^{1/2} v \in L^2(0, \infty)$. Then A_ϵ is defined by

$$A_\epsilon v(x) = (B + \epsilon x)\, v(x)$$

for all $v \in L^2(0, \infty)$ such that $xv \in L^2(0, \infty)$. Furthermore, since the topology induced by the quadratic form associated with $B + I$ is just the topology of H, $D(\mathfrak{A}) = D(A_\epsilon)$. For every $\epsilon > 0$, the spectrum of A_ϵ is $[0, \infty)$ and is purely continuous.

It now follows from the interpolation theorem of [5], pp. 431–432, that for $0 \leqslant \tau \leqslant 1$, V_τ consists of those functions $v \in L^2(0,\infty)$ such that $x^\tau v \in L^2(0, \infty)$ and

$$| v |_\tau \sim \left(\int_0^\infty x^{2\tau} \, | v(x)|^2 \, dx\right)^{1/2} + | v |$$

where "$\sim$" is read "is equivalent to". Hence if $0 \leqslant \tau < 1$ and

$$\int_0^\infty x^{2\tau} \, | k(x)|^2 \, dx < \infty,$$

then the spectrum of A_ϵ in J (an isolating interval for 1) is concentrated to order τ, while if

$$\int_0^\infty x^2 \, | k(x)|^2 \, dx < \infty$$

the spectrum of A_ϵ in J is concentrated to order p for all $p < 1$. The $\tau = 1/2$ case follows from Lemma 3.3, [1], while in the case $\tau = 1$ it follows from Lemma 3.2, [1], or Theorem 3.1, [2], that the spectrum of A_ϵ in J is concentrated to order 1.

References

1. Riddell, R. C., Spectral concentration for self-adjoint operators. *Pacific J. Math.* **23** (1967), 377–401.
2. Conley, C. C. and Rejto, P. A., Spectral concentration II—general theory, *in*, "Perturbation theory and its applications in quantum mechanics." pp. 129–143. (C. H. Wilcox, Ed). Wiley, New York, 1966.
3. Kato, T., On the convergence of the perturbation method. *J. Fac. Sci. Univ. Tokyo Sect. I.* **6** (1951), 145–226.
4. Kato, T., "Perturbation theory for linear operators." Springer Verlag, Berlin, 1966.
5. Lions, J. L., Espaces intermédiaires entre espaces hilbertiens et applications. *Bull. Math. Soc. Sci. Math. Phys., R. P. Roumaine* (N. S.) **2** (1958), 419–432.
6. Greenlee, W. M., Singular perturbation of eigenvalues. *Arch. Rational Mech. Anal.* (To appear).
7. Kato, T., Perturbation theory of semi-bounded operators. *Math. Ann.* **125** (1953), 435–447.
8. Rellich, F., Störungstheorie der Spektralzerlegung, III. *Math. Ann.* **116** (1939), 555–570.
9. Friedrichs, K. O. and Rejto, P. A., On a perturbation through which a discrete spectrum becomes continuous. *Commun. Pure Appl. Math.* **15** (1962), 219–235.